

A parametric variogram model bridging between stationary and intrinsically stationary processes

Martin Schlather
 Institut für Mathematik, Universität Mannheim
 A5, 6, D-68131 Mannheim
 schlather@math.uni-mannheim.de

8th December 2014

Abstract: A simple variogram model with two parameters is presented that includes the power variogram for the fractional Brownian motion, a modified De Wijsian model, the generalized Cauchy model and the multiquadrics model. One parameter controls the smoothness of the process. The other parameter allows for a smooth parametrization between stationary and intrinsically stationary second order processes in a Gaussian framework, or between mixing and non-ergodic max-stable processes when modeling spatial extremes by a Brown-Resnick process.

Keywords: Brown-Resnick process; Cauchy model; fractional Brownian motion; Gaussian process; variogram

1 Introduction

A Gaussian random field Z is a popular process to model spatial data in \mathbb{R}^d , mostly assuming some kind of at least underlying stationarity and isotropy of the field. A weakly stationary random field can be characterized by a bounded variogram γ , where $\gamma(h) = \frac{1}{2}E\{Z(h) - Z(0)\}^2$, $h \in \mathbb{R}^d$ (Gneiting and Guttorp, 2010). In contrast, the variogram of an intrinsically stationary random field might be unbounded. Although any unbounded variogram can be approximated arbitrarily closely on any compact set by a bounded variogram (Schlather and Gneiting, 2006), models that bridge between bounded and unbounded models seem to be used only rarely in statistics if at all, as appealing models are missing.

The Brown-Resnick process on \mathbb{R}^d (Brown and Resnick, 1977; Kabluchko et al., 2009) is a max-stable random field that is based on intrinsically stationary Gaussian random fields and is used to model spatial extreme values, such as heavy rainfall (Fawcett and Walshaw, 2014; Thibaud et al., 2013), avalanches (Blanchet and Davison, 2011), extreme temperature (Davison et al., 2013) or extreme wind (Zhang et al., 2014). If the underlying variogram model is bounded, then the corresponding Brown-Resnick field is not ergodic whilst a one-dimensional process has been shown to be mixing when the variogram model $\gamma(h)$, $h \in \mathbb{R}$, grows faster than $4 \log |h|$ (Wang and Stoev, 2010; Kabluchko and Schlather, 2010). Ergodicity might be assumed if the extremes are caused by local events, e.g., storm events, while non-ergodic models are suitable for spatially extended events such as cyclonic rainfall. When the kind of causative process for spatial extremes is not clear, a parametric model that allows for both ergodic and non-ergodic fields is advantageous.

For lack of awareness of suitable parametric models that bridge between stationary and intrinsically stationary Gaussian random fields, studies in both traditional geostatistics and spatial extreme value analysis using the Brown-Resnick model include two classes of models to cover the cases of bounded and unbounded variograms models, cf. Thibaud et al. (2013) or Wadsworth and Tawn (2014), for instance.

Here, we present an easily accessible, rotation invariant variogram model with two parameters that bridges between the class of bounded variograms and the class of unbounded ones. It includes several well-known models, but contains also novel parameter combinations. Hence, the model might be particularly useful in any application of spatial statistics.

2 The model

For $0 < \alpha \leq 2$ and $-\infty < \beta \leq 2$ let the function $\gamma_{\alpha,\beta} : \mathbb{R}^d \rightarrow [0, \infty)$ be defined as

$$\gamma_{\alpha,\beta}(h) = \frac{(1 + \|h\|^\alpha)^{\beta/\alpha} - 1}{2^{\beta/\alpha} - 1},$$

where the limiting function is $\log(1 + \|h\|^\alpha)/\log 2$ as $\beta \rightarrow 0$.

Proposition 1 *The function $\gamma_{\alpha,\beta}$ is a variogram in \mathbb{R}^d for any $d \in \mathbb{N}$, $0 < \alpha \leq 2$ and $-\infty < \beta \leq 2$.*

Proof. A positive constant and the identity being complete Bernstein functions, Corollary 7.12 and Proposition 7.10 in [Schilling et al. \(2010\)](#) yield that $f(\lambda) = (1 + \lambda^\delta)^{\varepsilon/\delta}$ is a Bernstein function for any $0 < \varepsilon \leq 1$ and $0 < |\delta| \leq 1$. Now $f(g)$ is a negative definite function for any negative definite function g and any Bernstein function f ([Schilling et al., 2010](#), Chapter 4). Choosing $g(h) = \|h\|^2$, $\varepsilon = \beta/2$ and $\delta = \alpha/2$, we get the assertion for positive values of β . For negative values of β we refer to the fact that $(1 + \gamma)^\beta$ is a positive definite function for any variogram γ , see ([Schlather, 2012](#), p. 36), for instance. By the basic properties of negative definite functions the limiting cases obtained by point-wise convergence are included.

The parameter α models the smoothness of both the variogram and the corresponding (Gaussian) random field ([Adler, 1981](#); [Scheuerer, 2010](#)) whereas β indicates the long range behavior ([Gneiting, 2000](#)). For negative values of β the model is bounded and heavy tailed, whereas for $0 \leq \beta \leq 2$ the variogram $\gamma_{\alpha,\beta}$ is unbounded. The normalizing factor is chosen such that $\gamma_{\alpha,\beta}(1) = 1$.

The model simplifies for $\beta = \alpha$ to the power model of a fractional Brownian random field ([Chilès and Delfiner, 1999](#)), for $0 < \beta \leq \alpha$ to a generalization of the fractional Brownian model described in [Gneiting \(2002\)](#) and [Schlather \(2010\)](#), and for $\beta < 0$ to the generalized Cauchy model ([Gneiting, 2000](#)). It also generalizes partially the multiquadrics and inverse multiquadrics used in approximation theory where $\alpha = 2$ and $\beta/\alpha \in \mathbb{R} \setminus \mathbb{N}_0$, see, for instance, [Buhmann \(1990\)](#), [Wendland \(2005\)](#) or [Lin and Yuan \(2006\)](#). As $\beta \rightarrow 0$, the limiting model equals a modified version of the De Wijsian model ([Wackernagel, 2003](#); [Matheron, 1962](#)). The case $0 < \alpha < \beta \leq 2$ is novel.

The presented variogram model can be generalized. Indeed, the proof of the proposition shows that $\gamma_{\delta,g} = \{(1 + g)^\delta - 1\}/(2^\delta - 1)$ is a variogram in \mathbb{R}^d for any variogram g and $-\infty < \delta \leq 1$ with limiting case $\gamma_{0,g} = \log(1 + g)/\log(2)$.

References

- R. Adler. *The Geometry of Random Fields*. John Wiley & Sons, Chichester, 1981.
- J. Blanchet and A.C. Davison. Spatial modelling of extreme snow depth. *Ann. Appl. Statist.*, 5: 1699–1725, 2011.
- B.M. Brown and S.I. Resnick. Extreme values of independent stochastic processes. *J. Appl. Probab.*, 14:732–739, 1977.
- MD Buhmann. Multivariate interpolation in odd-dimensional euclidean spaces using multiquadrics. *Constr. Approx.*, 6(1):21–34, 1990.
- J.-P. Chilès and P. Delfiner. *Geostatistics. Modeling Spatial Uncertainty*. John Wiley & Sons, New York, 1999.

- Anthony C Davison, Raphaël Huser, and Emeric Thibaud. Geostatistics of dependent and asymptotically independent extremes. *Math. Geosci.*, 45(5):511–529, 2013.
- L. Fawcett and D. Walshaw. Estimating the probability of simultaneous rainfall extremes within a region: a spatial approach. *J. Appl. Statist.*, 41:959–976, 2014.
- T. Gneiting. Power-law correlations, related models for long-range dependence and fast simulation. *J. Appl. Probab.*, 37:1104–1109, 2000.
- T. Gneiting. Nonseparable, stationary covariance functions for space-time data. *J. Amer. Statist. Assoc.*, 97:590–600, 2002.
- Tilmann Gneiting and Peter Guttorp. Continuous parameter stochastic process theory. In A.E. Gelfand, P.J. Diggle, M. Fuentes, and P. Guttorp, editors, *Handbook of Spatial Statistics*, chapter 2, pages 17–28. Chapman & Hall/CRC, 2010.
- Z. Kabluchko and M. Schlather. Ergodic properties of max-infinitely divisible processes. *Stochastic Process. Appl.*, 120:281–295, 2010.
- Z. Kabluchko, M. Schlather, and L. de Haan. Stationary max-stable fields associated to negative definite functions. *Ann. Probab.*, 37:2042–2065, 2009.
- Yi Lin and Ming Yuan. Convergence rates of compactly supported radial basis function regularization. *Statist. Sinica*, 16(2):425, 2006.
- G. Matheron. *Traité de Géostatistique Appliquée*, volume 1. Technip, Paris, 1962.
- M. Scheuerer. Regularity of the sample paths of a general second order random field. *Stochastic Process. Appl.*, 120:1879–1897, 2010.
- R. Schilling, R. Song, and Z. Vondracek. *Bernstein Functions*. de Gruyter, Berlin, 2010.
- M. Schlather. Some covariance models based on normal scale mixtures. *Bernoulli*, 16:780–797, 2010.
- M. Schlather. Construction of covariance functions and unconditional simulation of random fields. In E. Porcu, J.M. Montero, and M. Schlather, editors, *Advances and Challenges in Space-time Modelling of Natural Events*, Berlin, 2012. Springer.
- M. Schlather and T. Gneiting. Local approximation of variograms by covariance functions. *Statist. Probab. Lett.*, 76:1303–1304, 2006.
- E Thibaud, R Mutzner, and AC Davison. Threshold modeling of extreme spatial rainfall. *Water Resour. Res.*, 49(8):4633–4644, 2013.
- H. Wackernagel. *Multivariate Geostatistics*. Springer, Berlin, 3rd edition, 2003.
- Jennifer L Wadsworth and Jonathan A Tawn. Efficient inference for spatial extreme value processes associated to log-gaussian random functions. *Biometrika*, 101:1–15, 2014.
- Y. Wang and S.A. Stoev. On the structure and representations of max-stable processes. *Adv. Appl. Probab.*, 42:855–877, 2010.
- H. Wendland. *Scattered Data Approximation*. Cambridge University Press, Cambridge, 2005.
- Qiang Zhang, Mingzhong Xiao, Jianfeng Li, Vijay P Singh, and Zongzhi Wang. Topography-based spatial patterns of precipitation extremes in the poyang lake basin, china: Changing properties and causes. *J. Hydrol.*, 512:229–239, 2014.